OPTIMUM FORM OF CROSS SECTIONS OF PRISMATIC RODS WITH A LENGTHWISE CAVITY

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Several investigations have been devoted to determination of the optimum forms of the cross sections of isotropic prismatic rods. The following theorem was proposed in [1] and proved in [2]: With a specified cross-sectional area, a circular rod will have the greatest torsional rigidity. This result was generalized in [3] to the case of rods with a cavity: With known values of cross-sectional area and the area enveloped by the internal contour, a rod with a cross section in the form of a circular ring will have the greatest torsional rigidity.

The optimality condition was obtained in [4, 5] for the problem of determining the form of the cross section of a rod that would yield the greatest torsional rigidity for a given cross-sectional area. This made it possible, in turn, to solve the problem of finding the form of the cross section of a rod with a cavity so as to give the rod its maximum torsional rigidity in the case when the cross-sectional area is specified along with one of the boundary contours, which is not a circle [4-7].

Solutions were found in [8, 9] for problems concerning optimization of one of the parameters of a prismatic rod: cross-sectional area, torsional rigidity, or flexural rigidity, with limitations on the other two parameters. The problem of determining the form of the cross section of a rod to yield the maximum torsional rigidity with a given flexural rigidity was examined in [10, 11]. It was shown that the optimum rod cross sections in these problems are circular or elliptical.

Here we generalize the results in [8, 10, 11] to the case of rods with a doubly connected cross section.

1. We will examine the problem of determining the form of the cross section of an isotropic prismatic rod occupying a doubly-connected region Ω (see Fig. 1) that will yield the maximum torsional rigidity with specified axial moments of inertia for the cross section (flexural rigidities)

$$\iint_\Omega y^2 dxdy = J_x, \quad \iint_\Omega x^2 dxdy = J_y \tag{1.1}$$

and a fixed area F of the region D enveloped by the internal boundary contour L_1:

$$\iint_D dxdy = F. \tag{1.2}$$

We introduce a stress function during torsion $\varphi(x, y)$ [12] which satisfies the equation

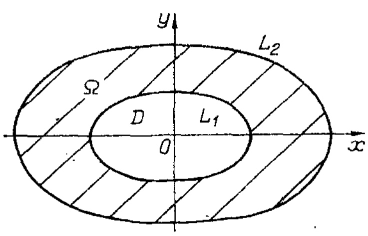

Fig. 1

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$$\varphi_{xx} + \varphi_{yy} + 2 = 0 \qquad (x, y) \in \Omega \tag{1.3}$$

and boundary conditions

$$\varphi = C(x, y) \in L_1, \; \varphi = 0 \qquad (x, y) \in L_2, \tag{1.4}$$

where the constant C is determined from the Bredt condition

$$2F = -\oint_{L_1} \varphi_n ds. \tag{1.5}$$

The derivative of $\varphi(x, y)$ with respect to a normal to the contour is designed through φ_n.

In accordance with [4, 5], the above-formulated optimization problem can be formulated as a variational problem on the stationary value of a functional

$$V = \iint_{\Omega} \left(4\varphi - \varphi_x^2 - \varphi_y^2 + \lambda_1 y^2 + \lambda_2 x^2\right) dxdy - \lambda_3 \iint_{D} dxdy \tag{1.6}$$

in the regions Ω and D with corresponding movable boundaries $L_1 + L_2$, L_1 in the case of boundary conditions (1.4).

Here, λ_1, λ_2, λ_3 are Lagrangian constant multipliers subject to determination.

With allowance for (1.4), the condition of stationariness of functional (1.6) corresponds to Eq. (1.3) and, due to the variation of L_1 and L_2, the optimality conditions determining the form of the boundary contours:

$$\begin{aligned} &\varphi_n^2 + \lambda_1 y^2 + \lambda_2 x^2 + \lambda_3 + 4C = 0 \qquad (x, y) \in L_1, \\ &\varphi_n^2 + \lambda_1 y^2 + \lambda_2 x^2 = 0 \qquad (x, y) \in L_2. \end{aligned} \tag{1.7}$$

Conditions (1.7) are satisfied for a rod cross section bounded by two geometrically similar ellipses. In fact, the stress function in torsion $\varphi(x, y)$ satisfying Eq. (1.3) for such a section has the form [13]

$$\varphi = -rx^2 + (r - 1)y^2 + t \qquad (x, y) \in L_1 + \Omega + L_2, \tag{1.8}$$

where $0 < r < 1$, $t > C > 0$ are constants. The stress function will satisfy boundary conditions (1.4) for a section with a boundary specified by the equations

$$\begin{aligned} &y^2 = [-rx^2 + (t - C)]/(1 - r) \qquad (x, y) \in L_1, \\ &y^2 = (-rx^2 + t)/(1 - r) \qquad (x, y) \in L_2, \end{aligned} \tag{1.9}$$

describing geometrically similar ellipses. The constant C, obtained from (1.5), has the form

$$C = t - F[r(1 - r)]^{1/2}/\pi. \tag{1.10}$$

In this case,

$$\begin{aligned} &\varphi_n^2 = 4\left[r(2r - 1)x^2 + (t - C)(1 - r)\right] \qquad (x, y) \in L_1, \\ &\varphi_n^2 = 4\left[r(2r - 1)x^2 + t(1 - r)\right] \qquad (x, y) \in L_2 \end{aligned}$$

and conditions (1.7) are satisfied as a result of unambiguous selection of the constants

$$\lambda_1 = -4(1 - r)^2, \; \lambda_2 = -4r^2, \; \lambda_3 = -4C.$$

The parameters r and t in solution (1.8)-(1.10) are determined by the well-known quantities of (1.1) and (1.2):

$$\begin{aligned} &r = \beta/(1 + \beta), \; t = R\beta^{1/2}/(1 + \beta), \\ &\beta = J_x/J_y, \; R = [4\pi(J_xJ_y)^{1/2} + F^2]^{1/2}/\pi. \end{aligned} \tag{1.11}$$

The torsional rigidity of the rod (to within the multiplier G, which is the shear stiffness of the material)

$$K = 2\left(\iint_{\Omega} \varphi dx dy + CF\right)$$

for an optimum cross section does not depend on the specified area of the hole F:

$$K_{\text{opt}} = 4J_x J_y/(J_x + J_y).$$

We use (1.9) and (1.11) to find the semiaxes of the external ellipse a and b along the axes Ox and Oy:

$$a = R^{1/2}\beta^{-1/4},\ b = R^{1/2}\beta^{1/4}. \tag{1.12}$$

The corresponding semiaxes of the internal contour cross section geometrically similar to the external section are equal to αa and αb, where $0 \leqq \alpha \leqq 1$ is a parameter equal to

$$\alpha = [F/(R\pi)]^{1/2}. \tag{1.13}$$

In accordance with (1.12), the ratio of the semiaxes of the boundary contours of the elliptical sections $a/b = \beta^{-\frac{1}{2}}$ depends only on the ratio of the required flexural rigidities.

Assuming that F = 0 in the resulting solution, i.e., that there is no lengthwise cavity in the rod, we arrive at the case of a singly-connected section: The maximum torsional rigidity for given flexural rigidities corresponds to a rod with a cross section bounded by an ellipse. This finding is in agreement with the theory proven in [10].

Let us examine the problem of determining the form of the cross section of a rod with a cavity on the condition that one of the axial moments of inertia – say J_x – be maximal for known values of the other moment J_y, torsional rigidity K, and the area F of the cross section of the cavity (1.2).

This problem is related to the problem examined previously. The optimality conditions determining the form of the boundary contours of the cross section differ from (1.7) only in the corresponding permutation of the Lagrangian multiplier λ_1, and the solution of the problem is satisfied by functions (1.8)-(1.10). The parameters r and t are expressed through the specified quantities J_y, K, and F, as follows:

$$\begin{gathered} r = 1/(1+\gamma^2),\ t = \gamma Q^{1/2}/[\pi(1+\gamma)], \\ \gamma = [(4J_y - K)/K]^{1/2},\ Q = 4\pi J_y/\gamma + F^2. \end{gathered} \tag{1.14}$$

We find from (1.14) that the existence of a solution requires that the given values of J_y and K satisfy the inequality

$$K < 4J_y. \tag{1.15}$$

If $J_y < J_x$, then $K < 4J_y$ for rods with a singly-connected cross section and, thus, also for rods with a cross section bounded by two geometrically similar ellipses [12]. Consequently, condition (1.15) is not an additional condition for the existence of the solution. Only an actual relationship between the specified values of K and J_y is required.

Thus, a cross section bounded by two geometrically similar ellipses is the form of the cross section of a rod with a cavity that will have the greatest flexural rigidity for given values of the other flexural rigidity and torsional rigidity satisfying the inequality (1.15) and a given value of the cross-sectional area of the cavity.

The semiaxes of the external boundary ellipse a and b, the ratio of the dimensions of the internal and external geometrically similar contours α, and the value of the moment J_x being optimized have the form

$$a = Q^{1/4}(\gamma/\pi)^{1/2},\ a/b = \gamma,\ \alpha = F^{1/2}/Q^{1/4},\ J_{x\,\text{opt}} = J_y/\gamma^2.$$

Assuming that F = 0, we arrive at the case of a solid rod: A rod with an elliptical cross section will have the greatest flexural rigidity for fixed values of the other flexural rigidity and torsional rigidity satisfying condition (1.15).

2. Let us examine the problem of determining the form of the cross section of a prismatic rod with a cavity having a minimum cross-sectional area

$$S = \iint_{\Omega} dxdy \tag{2.1}$$

with specified values of the axial moments of inertia (1.1) and the cross-sectional area of the cavity (1.2).

The optimality conditions have the form

$$1 + \lambda_1 y^2 + \lambda_2 x^2 + \lambda_3 = 0 \qquad (x,\ y) \in L_1,$$
$$1 + \lambda_1 y^2 + \lambda_2 x^2 = 0 \qquad (x,\ y) \in L_2,$$

where λ_1, λ_2, and λ_3 are constants subject to determination. The solution of the problem is satisfied by a rod cross section bounded by two geometrically similar ellipses. The parameters of the cross section are determined by the quantities J_x, J_y, and F from Eqs. (1.12)-(1.13).

The cross-sectional area of the optimum rod

$$S_{\text{opt}} = [4\pi(J_x J_y)^{1/2} + F^2]^{1/2} - F$$

with constant J_x and J_y decreases with an increase in the specified area of the hole F.

It should be noted that the optimum rods obtained correspond not only to a minimum cross-sectional area, but also to a maximum torsional rigidity with the same limitations (1.1)-(1.2).

The solution of the related optimization problem – the problem of determining the form of the cross section of a rod so as to maximize one of the axial moments of inertia, such as J_x, with specified values of S, J_y, and F – is also a cross section bounded by two geometrically similar ellipses:

$$\gamma = \frac{a}{b} = \frac{4\pi J_y}{S(S + 2F)},\ b = \frac{[S(S+F)(S+2F)]^{1/2}}{2\pi J_y^{1/2}},$$
$$\alpha^2 = F/(F+S),\ J_{x\,\text{opt}} = J_y/\gamma^2.$$

Assuming F = 0 in these problems, we arrive at the case of solid rods, and the optimum sections will be elliptical.

3. We will study the problem of determining the form of the cross section of a rod having a lengthwise cavity with the goal of minimizing the cross-sectional area S (2.1) with limitations on torsional rigidity K and on one of the axial moments of inertia (flexural rigidity), such as J_x:

$$K = 2\left(\iint_{\Omega} \varphi dxdy + CF\right) \geqslant K_0,\ J_x = \iint_{\Omega} y^2 dxdy \geqslant J_0 \tag{3.1}$$

and with a fixed area F enveloped by the internal contour L_1 of the cross section (see Fig. 1).

Following [8], we first examine the problem of minimizing S with only a limitation on torsional rigidity. Then a rod with a cross section in the form of a circular ring will have the smallest cross section with specified values of torsional rigidity $K = K_0$ and the area of the hole F. The radii of the ring

$$R_1 = (F/\pi)^{1/2},\ R_2 = [2K_0/\pi + (F/\pi)^2]^{1/4}. \tag{3.2}$$

The values of K and J_x for such a rod are connected by the relation $K = 2J_x$.

If the constants K_0 and J_0 in limitations (3.1) satisfy the inequality

$$K_0 \geqslant 2J_0, \tag{3.3}$$

then, as before, the optimum rod will be one with a cross section in the form of a circular ring in the problem of minimizing S with limitations (3.1) and a specified value of F, since in this case $K = K_0$, $J_x \geqq J_0$. The radii of the ring are determined by the parameters K_0 and F from equations (3.2).

If the specified parameters K_0 and J_0 violate condition (3.3) ($K_0 < 2J_0$), then both limitations (1.1) turn out to be significant in finding the optimum form of the cross section. Replacing limitations (3.1) by absolute equalities, we reduce the optimization problem to a variational problem on the stationary value of the functional

$$V = \iint_{\Omega} \left[1 + \lambda_1 \left(4\varphi - \varphi_x^2 - \varphi_y^2\right) + \lambda_2 y^2\right] dxdy - \lambda_3 \iint_{D} dxdy.$$

The optimality conditions obtained due to variation of the boundaries of the regions Ω and D have the form

$$\begin{aligned} &1 + \lambda_1 \varphi_n^2 + \lambda_2 y^2 + \lambda_3 + 4C\lambda_1 = 0 \qquad (x, y) \in L_1, \\ &1 + \lambda_1 \varphi_n^2 + \lambda_2 y^2 = 0 \qquad (x, y) \in L_2. \end{aligned} \tag{3.4}$$

Conditions (3.4) are satisfied for a rod cross section bounded by two geometrically similar ellipses. The solution of the problem is determined by the functions (1.8)-(1.10), where the parameters r and t depend on the quantities K_0, J_0, and F:

$$r = \frac{1}{1+\gamma^2}, \ t = \frac{\gamma \left(4\pi\gamma J_0 + F^2\right)^{1/2}}{\pi (1+\gamma^2)}, \ \gamma = \left(\frac{K_0}{4J_0 - K_0}\right)^{1/2}.$$

The semiaxes of the external boundary ellipse a and b, the ratio of the dimensions α of the internal and external geometrically similar contours, and the value of the optimum area of the cross section S have the form

$$\frac{a}{b} = \gamma, \ b = \frac{\left(4\pi\gamma J_0 + F^2\right)^{1/4}}{(\pi\gamma)^{1/2}},$$
$$\alpha = \frac{1}{b}\left(\frac{F}{\pi\gamma}\right)^{1/2}, \ S_{\text{opt}} = \left(4\pi\gamma J_0 + F^2\right)^{1/2} - F.$$

Two other optimization problems are related to the above-examined problem. The first consists of maximizing the torsional rigidity K of a rod having a doubly-connected cross section with limitations on the area (2.1) and on one of the axial moments of inertia of the cross section

$$S \leqslant S_0, \ J_x \geqslant J_0. \tag{3.5}$$

The second problem entails maximization of the flexural rigidity (axial moment of inertia J_x) with limitations on the cross-sectional area and torsional rigidity

$$S \leqslant S_0, \ K \geqslant K_0. \tag{3.6}$$

It is also assumed that in these problems we know the area F enveloped by the internal boundary contour of the cross section.

Let us return to the first problem. If we do not consider the limitation $J_x \geqq J_0$ in (3.5), then with specified values of the cross-sectional area S and the area of the hole F, the maximum torsional rigidity K will correspond to a rod cross section in the form of a circular ring [3]. The values of S and J_x for such a ring are connected by the relation

$$J_x = \frac{1+\alpha^2}{(1-\alpha^2)\, 4\pi} S^2, \ \alpha^2 = \frac{F}{S+F}.$$

If the specified values of S_0 and J_0 satisfy the inequality

$$J_0 \leqslant \frac{1+\alpha^2}{(1-\alpha^2)\, 4\pi} S_0^2, \tag{3.7}$$

then the optimum cross section will again be in the form of a circular ring (here, $S = S_0$, $J_x \geqq J_0$). The radii of the boundary circles $R_1 = (F/\pi)^{\frac{1}{2}}$, $R_2 = [(S_0 + F)/\pi]^{\frac{1}{2}}$.

If the parameters S_0 and J_0 do not satisfy Eq. (3.7), then determination of the optimum form of the cross section requires consideration of the limitation on flexural rigidity as well. It can be shown that the solution of the problem in this case is satisfied by a rod cross section bounded by two geometrically similar ellipses. Meanwhile, exact equalities are taken in (3.5). The parameters of the boundary ellipses a, b, and α and the torsional rigidity of the optimum rod are determined by the quantities J_0, S_0, and F:

$$\gamma = \frac{a}{b} = \frac{S_0(S_0+2F)}{4\pi J_0},\ b = 2\left[\frac{J_0(F+S_0)}{S_0(S_0+2F)}\right]^{1/2},\ \alpha^2 = \frac{F}{S_0+F},\ K_{\text{opt}} = 4J_0\frac{\gamma^2}{1+\gamma^2}.$$

Let us examine the second problem. Both limitations (3.6) are considered in the problem of maximizing flexural rigidity when determining the optimum form of the cross section. Replacing the inequalities in (3.6) by absolute equalities, we find that the solution of the problem is again satisfied by a rod with a cross section bounded by two geometrically similar ellipses:

$$\gamma = \frac{a}{b} = \frac{\delta}{2\pi}[1-(\delta^2-4\pi^2)^{1/2}],\ \delta = \frac{S_0(S_0+2F)}{K_0},$$
$$b = \left(\frac{S_0+F}{\pi\gamma}\right)^{1/2},\ \alpha^2 = \frac{F}{S_0+F},\ J_{x\text{opt}} = \frac{S_0(S_0+2F)}{4\pi\gamma}. \quad (3.8)$$

We find from (3.8) that the problem has a solution with the following limitation on the specified parameters K_0, S_0, and F:

$$K_0 \leqslant [S_0(S_0+2F)]/2\pi. \quad (3.9)$$

Limitation (3.9) has the following physical meaning: The specified torsional rigidity K_0 must not exceed the torsional rigidity of a rod with a cross section in the form of a circular ring having an area S_0 and a hole with an area F.

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